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Exercice 7

1. Mq $\mathcal{B}' = (1, X, X(X-1), X(X-1)(X-2))$ est une base de $\mathbb{C}_3[X]$

Posons: $P_0 = 1$
$P_1 = X$
$P_2 = X^2 - X$
$P_3 = X^3 - X^2 + 2X$

On a (P_0, P_1, P_2, P_3) une famille de 3 polynômes échelonnés en degré, donc libre, avec le $\underbrace{\text{bon nombre de polynômes}}_{\text{"}4 = \dim(\mathbb{R}_3[X])\text{"}}$, donc c'est bien une base de $\mathbb{C}_3[X]$. ✓

2. $$P_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ ✓

$$P_{\mathcal{B}'}^{\mathcal{B}} = \left(P_{\mathcal{B}}^{\mathcal{B}'}\right)^{-1}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 + 3L_4]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$ ✓

$$\xrightarrow[L_2 \leftarrow L_2 - 2L_4]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$ ✓

Rem: vous avez le droit de faire $L_3 \leftarrow L_3 + 3L_4$ et $L_2 \leftarrow L_2 - 2L_4$ d'un seul coup, car cela concerne 2 lignes différentes.

$$\xrightarrow[L_2 \leftarrow L_2 + L_3]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$ ✓

donc $$\boxed{P_{\mathcal{B}'}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$ ✓

3. Soit $Q = aX^3 + bX^2 + cX + d$. Ses coordonnées dans $\mathcal{B}$ forment le vecteur colonne $Y = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

En appliquant la formule de changement de base

$Y = P_{\mathcal{B}}^{\mathcal{B}'} Y'$ donc $Y' = P_{\mathcal{B}'}^{\mathcal{B}} Y$ ✓

Non car la base B est dans l'autre ordre:
$B = (1, X, X^2, X^3)$ donc $Y = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}$

$$Y' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \cancel{\begin{pmatrix} a \\ b+c+d \\ c+3d \\ d \end{pmatrix}}$$

intervertir $a \leftrightarrow d$, $c \leftrightarrow b$ pour obtenir le bon résultat

$$\boxed{Q = \underset{d}{\cancel{a}}P_0 + (\underset{c}{\cancel{b}}+\underset{b}{\cancel{c}}+\underset{a}{\cancel{d}})P_1 + (\underset{b}{\cancel{c}}+3\underset{a}{\cancel{d}})P_2 + \underset{a}{\cancel{d}}P_3}$$

Soit $A = \mathrm{Mat}_{\mathcal{B}}(u) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

D'après la formule de changement de base pour un endomorphisme

$A = PA'P^{-1} \iff AP = PA' \iff A' = P^{-1}AP$

$A' = P_{\mathcal{B}'}^{\mathcal{B}} A P_{\mathcal{B}}^{\mathcal{B}'}$

Calcul de $AP_{\mathcal{B}}^{\mathcal{B}'}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 & -2 \\ -1 & 0 & -1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 & -2 \\ -1 & -1 & -2 & 5 \\ 0 & -1 & -3 & 8 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

donc $\boxed{A' = \begin{pmatrix} 0 & -1 & 1 & -2 \\ -1 & -1 & -2 & 5 \\ 0 & -1 & -3 & 8 \\ 0 & 0 & 1 & 3 \end{pmatrix}}$

Exercice 20

1. $u_0 = 1$, $u_1 = 2$ et $u_{n+2} = \dfrac{u_{n+1}^6}{u_n^5}$

On pose $v_n = \ln(u_n)$

$$v_{n+2} = \ln(u_{n+2}) = \ln\left(\frac{u_{n+1}^6}{u_n^5}\right) = 6\ln(u_{n+1}) - 5\ln(u_n)$$

donc $v_{n+2} = 6v_{n+1} - 5v_n$ donc $v_{n+2} - 6v_{n+1} + 5v_n = 0$

E.C : $r^2 - 6r + 5 = 0$

$\Delta = (-6)^2 - 4 \times 5 = 36 - 20 = 16$

$x_{1,2} = \dfrac{6 \pm 4}{2} = 5$ ou 1 donc $\exists (A,B) \in \mathbb{R}^2 / \forall n \in \mathbb{N}$:

$v_n = A5^n + B \times 1^n$

$$\begin{cases} v_0 = A + B = \ln(u_0) = 0 \Rightarrow A = -B \\ v_1 = 5A + B = \ln(u_1) = \ln(2) \end{cases}$$

$$\begin{cases} A+B = 0 \\ 5A+B = \ln(2) \end{cases}$$

$$\underset{L_2 \leftarrow L_1 - 5L_1}{\Leftrightarrow} \begin{cases} A + B = 0 \\ -4B = \ln(2) \end{cases} \Leftrightarrow \begin{cases} A = \dfrac{\ln(2)}{4} \\ B = \dfrac{\ln(2)}{-4} \end{cases}$$

donc $v_n = \dfrac{5\ln(2)}{4} \times 5^n - \dfrac{\ln(2)}{4} \times 1^n$

$$v_n = \ln(2)\left(\frac{5^{n+1}}{4} - \frac{1^n}{4}\right)$$

donc $v_n = \dfrac{\ln(2)}{4}\left(5^{n+1} - 1^n\right)$ oui, ou 1 plus simplement

donc $\boxed{u_n = \exp\left(\dfrac{\ln(2)}{4}\left(5^{n+1} - 1^n\right)\right)}$

2. $u_0 = u_1 = 3$

$4u_{n+2} = 12u_{n+1} - 9u_n$

$\Rightarrow 4u_{n+2} - 12u_{n+1} + 9u_n = 0$

E.C : $4r^2 - 12r + 9 = 0$

$\Delta = (-12)^2 - 4 \times 4 \times 9 = 0$

$r = \frac{12}{8} = \frac{6}{4} = \frac{3}{2}$ ✓

$u_n = (A+Bn) \times \left(\frac{3}{2}\right)^n$

$u_0 = A + B \times 0 \times \left(\frac{3}{2}\right)^0 = A = 3$

$u_1 = (A+B) \times \frac{3}{2} = 3$

$A+B = 2 \Rightarrow B = 2-A$
$\Rightarrow B = -1$

donc $\boxed{u_n = (3-n) \times \left(\frac{3}{2}\right)^n}$ ✓

3. $\begin{cases} u_0 = 1 \\ u_1 = -1 \end{cases}$

$2u_{n+2} = u_{n+1} - u_n \Rightarrow 2u_{n+2} - u_{n+1} + u_n = 0$

$2r^2 - r + 1 = 0$

$\Delta = (-1)^2 - 4 \times 2 \times 1 = -7$

$r_{1,2} = \frac{1 \pm i\sqrt{7}}{4}$ ✓

$u_n = \rho^n (A\cos(n\theta) + B\sin(n\theta))$

$\rho = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2} = \frac{1}{\sqrt{2}}$ ✓

$u_n = \left(\frac{1}{\sqrt{2}}\right)^n (A\cos(n\theta) + B\sin(n\theta))$ ✓

$u_0 = A = 1$ ✓

$u_1 = -1$ et $u_1 = \frac{1}{\sqrt{2}}(\cos(\theta) + B\sin(\theta))$

or $\cos(\theta) = \frac{\frac{1}{4}}{\frac{1}{\sqrt{2}}}$ ✓ et $\boxed{\sin(\theta) = \frac{\frac{\sqrt{7}}{4}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{14}}{4}}$ ✓

$\boxed{\cos(\theta) = \frac{\sqrt{2}}{4}}$ ✓

$u_1 = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{4} + B \cdot \frac{\sqrt{14}}{4}\right)$ ✓

$= \frac{1}{4} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{14}}{4} \cdot B$

$= \frac{1}{4} + B\frac{\sqrt{7}}{4} = -1$

$\Rightarrow \quad B\dfrac{\sqrt{7}}{4} = -\dfrac{5}{4} \qquad \Rightarrow \quad B = -\dfrac{5}{\sqrt{7}}$

donc $\boxed{u_n = \left(\dfrac{1}{\sqrt{2}}\right)^n \left(\cos(n\theta) - \dfrac{5}{\sqrt{7}} \sin(n\theta)\right)}$

4. $\forall n \in \mathbb{N},\ u_{n+2} = 2u_{n+1} + 8u_n + 9n^2$

$u_{n+2} - 2u_{n+1} - 8u_n = 9n^2$

ESSMA: $u_{n+2} - 2u_{n+1} - 8u_n = 0$

$r^2 - 2r - 8 = 0$

$\Delta = (-2)^2 - 4 \times 1 \times (-8) = 4 + 32 = 36$

$r_{1,2} = \dfrac{2 \pm 6}{2} = 4$ ou -2 donc $\exists (A, B) \in \mathbb{R}^2 / \forall n \in \mathbb{N}$

$u_{n,h} = A4^n + B(-2)^n$

$u_{n,p} = an^2 + bn + c$

Comme on a : $u_{n+2} - 2u_{n+1} - 8u_n = 9n^2$

On injecte: $a(n+2)^2 + b(n+2) + c$

$- 2(a(n+1)^2 + b(n+1) + c)$

$- 8(an^2 + bn + c) = 9n^2$

$a(n^2 + 4n + 4) + b(n+2) + c - 2(a(n^2 + 2n + 1) + b(n+1) + c) - 8(an^2 + bn + c) = 9n^2$

$an^2 + 4an + 4a + bn + 2b + c - 2an^2 - 4na - 2a - 2bn - 2b - 2c - 8an^2 - 8bn - 8c = 9n^2$

donc $n^2(a - 2a - 8a) + n(4a + b - 4a - 2b - 8b) + 2a - 9c = 9n^2$

$-9an^2 - 9bn + 2a - 9c = 9n^2$

$$\begin{cases} -9a = 9 \\ -9b = 0 \\ 2a - 9c = 0 \end{cases} \iff \begin{cases} a = -1 \\ -9b = 0 \Rightarrow b = 0 \\ -9c = -2a \end{cases}$$

$$\iff \begin{cases} a = -1 \\ b = 0 \\ c = -\dfrac{2}{-9}a \end{cases}$$

$$\iff \begin{cases} a = -1 \\ b = 0 \\ c = -\dfrac{2}{9} \end{cases}$$

donc $u_n = A \times 4^n + B \times (-2)^n - n^2 - \frac{2}{9}$

$u_0 = A + B - \frac{2}{9} = 0$

$u_1 = 4A - 2B - 1 - \frac{2}{9} = 1$

On a : $\begin{cases} A+B = \frac{2}{9} \\ 4A - 2B = \frac{20}{9} \end{cases} \underset{L_2 \leftarrow L_2 - 4L_1}{\iff} \begin{cases} A+B = \frac{2}{9} \\ -6B = \frac{20}{9} - \frac{8}{9} = \frac{12}{9} \end{cases}$

$\iff \begin{cases} A+B = \frac{2}{9} \\ B = -\frac{2}{9} \end{cases}$

$\iff \begin{cases} A = \frac{4}{9} \\ B = -\frac{2}{9} \end{cases}$

Finalement : $\boxed{u_n = \frac{4}{9} \times 4^n + \left(-\frac{2}{9}\right) \times (-2)^n - n^2 - \frac{2}{9}}$