Exercice 2 :

Choisis $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ plutôt.

1) On note $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$, $\ker(f) = \ker(A) = \{X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid AX = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

On a $AX = 0 \iff \begin{pmatrix} x_1 + 2x_3 \\ -x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} x_1 + 2x_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{cases}$

$$\underset{L_2 \leftarrow L_2 + L_1}{\iff} \begin{cases} x_1 + 2x_3 = 0 \\ x_2 + 3x_3 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 = -2x_3 \\ x_2 = -3x_3 \end{cases}$$

Donc $\ker(A) = \{\begin{pmatrix} -2x \\ -3x \\ x \end{pmatrix} \mid x \in \mathbb{R}\} = \text{vect}(\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix})$ ✓

Ainsi $\boxed{\ker(f) = \text{vect}(X^2 - 3X - 2)}$ ✓ donc $\dim(\ker(f)) = \dim(\ker(A)) = 1$ ✓

D'après le théorème du rang : $3 = \dim(\ker(A)) + \dim(\text{Im}(A))$

donc $\dim(\text{Im}(A)) = 3 - \dim(\ker(A))$

$= 3 - 1$

$= 2$ ✓

Ainsi $\dim(\text{Im}(f)) = \dim(\text{Im}(A)) = 2$ on a $\text{Im}(f)$ sous-ev de $\mathbb{R}^2$ et de même dimension

Donc $\boxed{\text{Im}(f) = \mathbb{R}^2}$ ✓

prends $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ plutôt.

2) $\begin{pmatrix} 2 & 2 & -1 & -1 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & 1 & -2 \\ 1 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2x_1 + 2x_2 - x_3 - x_4 = 0 \\ 2x_1 + 2x_2 - x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 - 2x_4 = 0 \\ x_1 + x_2 - 2x_3 + x_4 = 0 \end{cases}$

$$\underset{\substack{L_3 \leftarrow 2L_3 - L_1 \\ L_4 \leftarrow 2L_4 - L_1}}{\iff} \begin{cases} 2x_1 + 2x_2 - x_3 - x_4 = 0 \\ +3x_3 - 3x_4 = 0 \\ -3x_3 + 3x_4 = 0 \end{cases}$$ ✓ ✓

$$\iff \begin{cases} 2x_1 + 2x_2 - x_3 - x_4 = 0 \\ x_3 = x_4 \end{cases}$$ ✓ ✓

$$\iff \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_3 = x_4 \end{cases}$$ ✓ ✓

$$\iff \begin{cases} x_1 = -x_2 + x_3 = x_4 \\ x_3 = x_4 \end{cases}$$ ✓ ✓

Donc $\boxed{\ker(A)} = \{\begin{pmatrix} -y + x \\ y \\ x \\ x \end{pmatrix}, (x,y) \in \mathbb{R}^2\} = \boxed{\text{vect}(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix})}$ ✓

Déterminons $\text{Im}(A)$:

D'après le théorème du rang : $4 = \dim(\ker(A)) + \dim(\mathrm{Im}(A))$

Or $\dim(\ker(A)) = 2$

Donc $\dim(\mathrm{Im}(A)) = \dim(A) - \dim(\ker(A))$

$= 4 - 2$

$\dim(\mathrm{Im}(A)) = 2$

En appelant C_1, C_2, C_3, C_4 les colonnes de A, on remarque que $\mathrm{Im}(A) = \mathrm{vect}(C_1, C_2, C_3, C_4)$ et

$C_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ainsi $C_1 = C_2$

$C_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $C_4 = \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}$, $-C_3 - C_1 = \begin{pmatrix} +1-2 \\ +1-2 \\ -1-1 \\ +2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix} = C_4$

Donc $\boxed{\mathrm{Im}(A)} = \mathrm{vect}(C_1 ; C_2 ; C_3 ; -C_3 - C_1) = \mathrm{vect}(C_1, C_3) = \boxed{\mathrm{vect}\left(\begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -2 \end{pmatrix}\right)}$

On cherche $\ker(A) \cap \mathrm{Im}(A)$:

Soit $M \in \ker(A) \cap \mathrm{Im}(A)$ alors $\exists (x, y, z, t) \in \mathbb{R}^4 \,/\, M = x\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = z\begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \\ -2 \end{pmatrix}$

$$\begin{cases} x - y - 2z + t = 0 \\ + y - 2z + t = 0 \\ x - z - t = 0 \\ x - z + 2t = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - y - 2z + t = 0 \\ +y - 2z + t = 0 \\ +y + z - 2t = 0 & L_3 \leftarrow L_3 - L_1 \\ +y + z + t = 0 & L_4 \leftarrow L_4 - L_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - y - 2z + t = 0 \\ +y - 2z + t = 0 \\ +3z - 3t = 0 & L_3 \leftarrow L_3 - L_2 \\ +3z = 0 & L_4 \leftarrow L_4 - L_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - y + t - 2z = 0 \\ y + t - 2z = 0 \\ -3t + 3z = 0 \\ 3z = 0 \end{cases}$$

Ce système est de Cramer, il y a donc une unique solution qui est $M = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Donc $\boxed{\mathrm{Im}(A) \cap \ker(A) = \{0_{4,1}\}}$

Ainsi la somme de $\ker(A) + \mathrm{Im}(A)$ est directe, de plus $\dim(\mathrm{Im}(A) \oplus \ker(A)) = \dim(\mathrm{Im}(A)) + \dim(\ker(A))$

$= 2 + 2 = 4$

On a donc $\mathrm{Im}(A) \oplus \ker(A)$ sous-ev de $\mathbb{R}^4$ de dimension $4 = \dim(\mathbb{R}^4)$

T.B.

Ainsi $\boxed{\mathrm{Im}(A) \oplus \ker(A) = \mathbb{R}^4}$